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A topological string: the Rasetti–Regge Lagrangian, topological quantum field theory and vortices in quantum fluids

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Abstract

The kinetic part of the Rasetti–Regge action $I_{\rm RR}$ for vortex lines is studied and its relevance to string theory is established. It is shown that both $I_{\rm RR}$ and the Polyakov string action $I_{\rm Pol}$ can be constructed with the same field X^μ. Unlike $I_{\rm NG}$, however, $I_{\rm RR}$ describes a Schwarz-type topological quantum field theory. Using generators of classical Lie algebras, $I_{\rm RR}$ is generalized to higher dimensions. In all dimensions, the momentum 1-form $\mathbf{P}$ constructed from the canonical momentum for the vortex belongs to the first cohomology class $H^1(M, \mathbb{R}^m)$ of the worldsheet M swept out by the vortex line. The dynamics of the vortex line thus depend directly on the topology of M. For a vortex ring, the equations of motion reduce to the Serret–Frenet equations in $\mathbb{R}^3$, and in higher dimensions they reduce to the Maurer–Cartan equations for $so(m)$.

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1. Introduction

The fact that vortex lines play an important role in many physical systems is well known [1, 2]. Following the original ideas of Onsager [3] and Feynman [4], researchers have even used vortex rings in an attempt to explain the underlying cause of the lambda transition for superfluid He^4 [5, 6]. More recently, the discovery of Bose–Einstein condensates (BEC) [7, 8] has renewed interest in the study of vortex lines, and during the last year vortex excitations have been observed experimentally in BEC [9–12]. There has been a corresponding theoretical interest in the formation and stability of vortex lines in BEC (see, for example, [13, 14]).

While experimental studies of vortex lines in quantum fluids have been remarkable, theoretical understanding of the dynamics and interactions of vortex lines on a quantum level

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has proceeded at a much slower pace. Recent theoretical work on vortices in BEC has mostly been focused on the *formation* and *stability* of vortices in the condensate, and not on the properties and dynamics of the vortices once they are formed.

Much of the efforts in developing a deeper understanding of the dynamics of vortex lines on a quantum level are based on the work of Rasetti and Regge. Using arguments from classical fluid dynamics of ideal fluids, they [15] proposed a Lagrangian for studying the quantum theory of vortex lines in quantum fluids in three dimensions. Vortex lines are treated as extended objects and like a string in string theory, a vortex line sweeps out a two-dimensional worldsheet M as it propagates in time. Current algebra methods leading to the study of $Sdiff(\mathbb{R}^3)$, the diffeomorphic group in $\mathbb{R}^3$, were then used by them and subsequent researchers [16–23] in an effort to quantize the field.

In this paper we propose a different approach to understand the dynamics of vortex lines. Focusing on a single vortex, we start with the field X^μ and make use of the $so(3)$ Lie algebra to rewrite the kinetic part of the full Rasetti–Regge action I_{RR} in terms of differential forms, and demonstrate how I_{RR} is related to the Polyakov form [24] of the Nambu–Goto action I_{Pol} [25, 26] (see also [27] for a different approach). It is then straightforward to see that while I_{Pol} defines a propagating string, I_{RR} defines, when quantized, a Schwarz-type topological quantum field theory (TQFT) [28–30]. Indeed, I_{RR} is very similar in form to the Chern–Simons Lagrangian. Making use of other classical Lie algebras, we then extend this construction to higher dimensions; the linkage between I_{Pol} and I_{RR} still holds (see also [31]). However, unlike I_{Pol}, which can be constructed in any dimension, I_{RR} exists only in a discrete number of dimensions corresponding to the dimension of the Lie algebra used in its construction.

Using this approach, it becomes clear that the understanding of the quantum—and thus statistical behaviour—of vortex lines will be the first real-world application of TQFT. Conversely, the vortex system provides a means of studying experimentally a TQFT for the first time. The purpose of this paper is thus to establish the relation between I_{RR} and TQFT. Our approach is strictly classical, and our analysis formal. Nonetheless, using this classical analysis and the fact that I_{RR} is a TQFT, a great deal can immediately be discerned about the properties of vortex lines.

As is well known, a TQFT does not define a dynamical system in a traditional sense; a single-vortex line does not, strictly speaking, have dynamical variables that evolve with time. TQFTs are interesting nonetheless [28, 29]. While our approach is strictly classical, even at this level we find deep correlation between the topology and the dynamics of vortices. Indeed, we show that the momentum 1-form $\mathbf{P}$ constructed from the canonical momentum of the vortex line belongs to the first cohomology class of M; the dynamics of vortices depend directly on the topology of M. Going further, we show formally that the solution to the equations of motion in three dimensions reduces to the Serret–Frenet equations for arbitrary curves in $\mathbb{R}^3$. These equations are themselves equivalent to the equation of motion of a charged particle constrained to move on a unit sphere in the presence of a dyon located at the centre. In higher dimensions, the equations of motion reduce to the Maurer–Cartan equations for $so(m)$. The Maurer–Cartan 1-forms can be interpreted as a 'pure gauge' non-Abelian vector potential, and as is the case for TQFT, we are working with flat vector bundles. With this analogy, explicit solutions of the equations of motion can be found using Wilson path ordering.

Correlation between I_{RR} and string theory go beyond the construction of I_{Pol}, however. A term of the form $\int B_{\mu\nu}\, dX^\mu \wedge dX^\nu$, where $B_{\mu\nu}$ is an antisymmetric tensor functional of the string field, was added to I_{Pol} by Callan *et al* [32] in their background-field treatment of string. $B_{\mu\nu}$ generates an all-pervasive magnetic field in spacetime. While similar in form to I_{RR}, in their treatment the specific functional dependence of $B_{\mu\nu}$ on X^μ was determined by requiring that the trace anomaly of the *total* string action vanish. This resulted in a $B_{\mu\nu}$ that is dramatically different from what is considered here. Along similar lines, Giveon *et al* [33] also considered a $B_{\mu\nu}$ term in the string Lagrangian, but relaxed the trace anomaly condition and considered the effect of *constant* $B_{\mu\nu}$ on the string.

2. General construction of I_{RR}

We begin with a classical, real Lie algebra $\mathfrak{g}$ with generators $\mathbf{T}_a$ such that $[\mathbf{T}_a, \mathbf{T}_b] = f_{ab}{}^c\mathbf{T}_c$, where $f_{ab}{}^c$ are the structure constants for $\mathfrak{g}$, and indices run from 1 to m, the dimension of $\mathfrak{g}$. $\mathbf{T}^a$ is represented by matrices and following the convention in [34], the Killing form $h_{ab} \equiv \mathrm{Tr}\{\mathbf{T}_a\mathbf{T}_b\} = -\delta_{ab}$ is used to raise and lower the indices: $A_a = h_{ab}A^b = -A^b$. With this orthonormality condition, we can use the set $\{\mathbf{T}_a\}$ as a natural basis for $\mathbb{R}^m$, with $\mathbf{V} \in \mathbb{R}^m$ given by $\mathbf{V} = V^a\mathbf{T}_a$. The inner product on $\mathbb{R}^m$ is then $\langle\mathbf{V}, \mathbf{U}\rangle \equiv -\mathrm{Tr}\{\mathbf{VU}\}$ for $\mathbf{V}, \mathbf{U} \in \mathbb{R}^m$. Furthermore, using the identity matrix $\mathbf{I}$ of $\mathfrak{g}$ we can extend this construction to the $(m + 1)$-dimensional Minkowski space Min by taking $V = V_0\mathbf{I}/\sqrt{m} + \mathbf{V}$ for $V \in$ Min. When $\mathfrak{g} = su(2)$, this is just a representation of Minkowski space by the quarternions.

We next consider an extended object $X^\mu(x^0, x^1)$ sweeping out a 2D surface M in Min where x^0 and x^1 are the spatial and time coordinates on M. Taking $X = X_0\mathbf{I}/\sqrt{m} + X_a\mathbf{T}^a$, the usual Nambu–Goto string action is obtained through

$$I_{\mathrm{Pol}} \equiv -\mathrm{Tr}\int \mathrm{d}X \wedge *\,\mathrm{d}X$$
$$= \int \sqrt{-g}g^{AB}\eta_{\mu\nu}\partial_A X^\mu\partial_B X^\nu\,\mathrm{d}x^0\,\mathrm{d}x^1 \qquad (1)$$

where capital roman indices run from 0 to 1, $\mathrm{d} = \mathrm{d}x^A\partial_A$ is the exterior derivative on M, $*$ is the Hodge $*$-operator and g_{AB} is the worldsheet metric. In this case X^μ describes a string.

The kinetic part of the Rasetti–Regge action is also constructed from X, but now

$$I_{\mathrm{RR}} \equiv -\frac{1}{3}\,\mathrm{Tr}\int X\,\mathrm{d}X \wedge \mathrm{d}X = -\frac{1}{3}\,\mathrm{Tr}\int \mathbf{X}\,\mathrm{d}\mathbf{X} \wedge \mathrm{d}\mathbf{X}$$
$$= -\frac{1}{3}\int f_{abc}X^a\partial_0X^b\partial_1X^c\,\mathrm{d}x^0\,\mathrm{d}x^1. \qquad (2)$$

X^μ in this case describes a vortex line. Note, however, that g_{AB} does not explicitly appear; I_{RR} is a topological invariant and describes a Schwarz-type TQFT similar to Chern–Simons theory. (This corresponds to an antisymmetric-field Lagrangian in background-field string theory with $B_{ab} \sim \epsilon_{abc}X^c$ in three dimensions.) Note also that I_{RR} is translationally invariant; the Lagrangian changes by a total derivative, $X\,\mathrm{d}X\wedge\mathrm{d}X \to X\,\mathrm{d}X\wedge\mathrm{d}X + \Xi\,\mathrm{d}(X\,\mathrm{d}X)$ under the uniform translation $X \to X + \Xi$. Indeed, I_{RR} is the *only* translationally invariant topological action that can be constructed directly from X^μ. In the special case of $\mathfrak{g} = so(3)$, I_{RR} is proportional to the Lagrangian in [15], but without the coupling due to self-interaction.

Note that I_{RR} does not depend on X^0, the time component of X^μ. This is expected: topological Lagrangians describe systems with no dynamical degrees of freedom. We will thus work solely with $\mathbf{X}$ from this point. This $\mathbf{X}$ is a section of the vector bundle $\mathbb{R}^m$ over M, and is at the same time an element of $\mathfrak{g}$, a *vector* on $\mathbb{R}^m$ and a 0-form (and thus a function) on M. Therefore, the structure constants f_{abc} form a rank-3, totally antisymmetric tensor on $\mathbb{R}^m$. The 1-form $\mathbf{F} = \mathbf{F}_A\,\mathrm{d}x^A = F^a_A\,\mathrm{d}x^A\mathbf{T}_a$ is then a *vector-valued* or, equivalently, a Lie algebra-valued 1-form on M, meaning that each of its two *components* $\mathbf{F}_A$ is both vector in $\mathbb{R}^m$ and member of $\mathfrak{g}$.

The equations of motion, $\mathrm{d}\mathbf{X}\wedge\mathrm{d}\mathbf{X} = 0$, from equation (2) can be integrated once to give

$$\mathbf{P} \equiv [\mathbf{X},\ \mathrm{d}\mathbf{X}] \qquad (3)$$

where $\mathbf{P}$ is a *closed* Lie algebra-valued 1-form on the worldsheet: $\mathrm{d}\mathbf{P} = 0$. The two components of $\mathbf{P}$ are $\mathbf{P}_0 \equiv P_0^c \mathbf{T}_c = f_{ab}{}^c X^a \partial_0 X^b \mathbf{T}_c$ and $\mathbf{P}_1 \equiv P_1^c \mathbf{T}_c = f_{ab}{}^c X^a \partial_1 X^b \mathbf{T}_c$. $\mathbf{P}$ is related to the canonical momentum for $\mathbf{X}$ through the dual form $\mathbf{\Pi} \equiv *\mathbf{P}$

$$\Pi_a^A = \frac{1}{\sqrt{-g}} \frac{\delta I_{\mathrm{RR}}}{\delta \partial_A X^a} \tag{4}$$

where $\mathbf{\Pi} = \Pi_A^a \, \mathrm{d}x^A \mathbf{T}_a$. The components of $\mathbf{P}$ then determine the momentum of the vortex, and we call $\mathbf{P}$ the *momentum 1-form*.

This choice for $\mathbf{P}$ is only unique up to a total derivative. Although we could have just as well chosen $\mathbf{P}' = 2\mathbf{X}\,\mathrm{d}\mathbf{X}$, $\mathbf{P} - \mathbf{P}' = \mathrm{d}\mathbf{X}^2$, and these two choices differ by an exact form. Indeed, under uniform translations, $\mathbf{X} \to \mathbf{X} + \mathbf{K}$, $\mathbf{P} \to \mathbf{P} + \mathrm{d}[\mathbf{K}, \mathbf{X}]$, and $\mathbf{P}$ changes by an *exact* 1-form. Conversely, suppose we have $\mathbf{P}_1 \equiv [\mathbf{X}_1, \mathrm{d}\mathbf{X}_1]$ and $\mathbf{P}_2 \equiv [\mathbf{X}_2, \mathrm{d}\mathbf{X}_2]$ that differ by a close form $\mathrm{d}\mathbf{F}$. Then $\mathrm{d}\mathbf{F} = \mathrm{d}[\mathbf{X}_2 - \mathbf{X}_1, \mathbf{X}_2 + \mathbf{X}_1]/2 - [\mathbf{X}_2 + \mathbf{X}_1, \mathrm{d}(\mathbf{X}_2 - \mathbf{X}_1)]$ so that either $\mathbf{X}_2 - \mathbf{X}_1 = \mathbf{K}$ or $\mathbf{X}_2 + \mathbf{X}_1 = \mathbf{K}$, where $\mathbf{K}$ is a constant. Thus, $\mathbf{X}_2$ is related to $\mathbf{X}_1$ by a uniform translation or a reflection plus a translation. Therefore, what is physically relevant are the *equivalence classes* of $\mathbf{P}$, where $\mathbf{P}_1 \sim \mathbf{P}_2$ if they differ by an exact form, and not any one specific choice of $\mathbf{P}$. Consequently, $\mathbf{P} \in H^1(M, \mathbb{R}^m)$, the first cohomology class of M, and we are interested in $\mathbf{P}$ that are closed but *not* exact.

The cohomology classes for 2D surfaces are well known [35]. In particular, $H^1(M, \mathbb{R})^m = 0$ if M is *not* a closed surface. This result has definite implications for the dynamics of vortex lines: the dynamics of an open vortex line, which sweeps out a 2D open sheet in $\mathbb{R}^m$, differ dramatically from that of a closed vortex line (vortex ring), which sweeps out a closed surface.

For the open vortex line, $H^1(M, \mathbb{R}^m) = 0$ and we can always make a translation to a frame in which the momentum vanishes, $\mathbf{P} = 0$ so that $0 = [\mathbf{X}, \mathrm{d}\mathbf{X}]$. The solution for $\mathbf{X}$ in this case is particularly simple. For $\mathfrak{g} = so(3), su(2), sp(2)$, $\mathbf{X} = a(x^0, x^1)\mathbf{H}$, where a is an arbitrary function and $\mathbf{H}$ is a constant vector. The vortex line is constrained to move along one direction: $\mathbf{H}$. For other Lie algebras, $\mathbf{X} \in \mathfrak{c}$, the Cartan subalgebra for $\mathfrak{g}$, so that $\mathbf{X} = X^i \mathbf{H}^i$ where $\{\mathbf{H}^i\}$ form the bases for $\mathfrak{c}$ [34], and $\mathbf{X}$ propagates within a linear subspace of $\mathbb{R}^m$.

For the vortex ring, on the other hand, $H^1(M, \mathbb{R}^m) = \mathbb{Z}^m$, the integers, and $\mathbf{P}$ need not vanish. The dynamics of vortex rings are thus much more interesting, and we shall focus on them for the rest of the paper. We begin with $\mathfrak{g} = so(3)$, $su(2)$ or $sp(2)$. The vortex is propagating in $\mathbb{R}^3$ and its dynamics are especially constrained.

3. Vortex rings in $\mathbb{R}^3$

When $\mathfrak{g} = so(3)$, $f_{abc} = \epsilon_{abc}$ and we deal with a vortex line propagating in $\mathbb{R}^3$. Although we can revert to the usual vector notation in this case, doing so will add notational complexity. Instead, we introduce a slight abuse of notation and write the cross product of two vectors $\mathbf{V}, \mathbf{U} \in \mathbb{R}^3$ as $\mathbf{V} \times \mathbf{U} \equiv [\mathbf{V}, \mathbf{U}]$.

It is straightforward to show that $[\mathbf{P}_0, \mathbf{P}_1] = 0$; the two vectors are proportional to one another. Consequently, we can write $\mathbf{P} = \hat{\mathbf{b}} p$ where p is a scalar 1-form on M and $\hat{\mathbf{b}} \in \mathbb{R}^3$. (The hat denotes a unit vector: $|\hat{\mathbf{b}}|^2 = \langle \hat{\mathbf{b}}, \hat{\mathbf{b}} \rangle = 1$.) Because $\mathrm{d}\mathbf{P} = 0$, $\mathrm{d}\hat{\mathbf{b}} \wedge p + \hat{\mathbf{b}}\,\mathrm{d}p = 0$; each term must vanish separately. Consequently, $\mathrm{d}p = 0$ and p is a closed 1-form. For the other term, $\mathrm{d}\hat{\mathbf{b}} \wedge p = 0$, and from Cartan's lemma [36], $\mathrm{d}\hat{\mathbf{b}}$ must be proportional to p: $\mathrm{d}\hat{\mathbf{b}} = -\tau \hat{\mathbf{n}} p$, where τ is an arbitrary function and $\hat{\mathbf{n}}$ is a unit vector in $\mathbb{R}^3$ orthogonal to $\hat{\mathbf{b}}$. Doing this trick once again and noting that $\mathrm{dd}\hat{\mathbf{b}} = 0$, $\mathrm{d}\hat{\mathbf{n}} = (-\kappa \hat{\mathbf{t}} + \tau \hat{\mathbf{b}}) p$ where κ is another arbitrary function on M and $\hat{\mathbf{t}} = \hat{\mathbf{n}} \times \hat{\mathbf{b}}$. Once again $\langle \hat{\mathbf{n}}, \hat{\mathbf{t}} \rangle = 0$. It is then straightforward to show that $\mathrm{d}\hat{\mathbf{t}} = \kappa \hat{\mathbf{n}}$, and no more terms need to be introduced.

To complete the solution for $\mathbf{X}$, we note that $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{b}}$ form a moving orthogonal coordinate system on $\mathbb{R}^3$. Taking $\mathbf{X} = |\mathbf{X}|(\alpha\hat{\mathbf{t}} + \beta\hat{\mathbf{n}} + \gamma\hat{\mathbf{b}})$ for constants α, β, γ, we require that this $\mathbf{X}$ solves equation (3). Then $\alpha = 1$ and $\mathbf{X}$ lies along $\hat{\mathbf{t}}$, while $|\mathbf{X}|^2 = 1/\kappa$. Solution of equations of motion therefore reduces to finding solutions for $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ for given κ, τ and p.

From $\langle\hat{\mathbf{b}}, \hat{\mathbf{n}}\rangle = 0$ and $\mathrm{d}\hat{\mathbf{t}} = \kappa\hat{\mathbf{n}}$, we see that $\mathrm{d}\tau$ and $\mathrm{d}\kappa$ are both proportional to the 1-form p; the functions κ, τ that determine $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{b}}$ all depend upon p. Consequently, there is a function $s(x^0, x^1)$ such that *locally* $\mathrm{d}s = p$ and

$$\hat{\mathbf{t}}' = \kappa(s)\hat{\mathbf{n}} \qquad \hat{\mathbf{n}}' = -\kappa(s)\hat{\mathbf{t}} + \tau(s)\hat{\mathbf{b}} \qquad \hat{\mathbf{b}}' = -\tau(s)\hat{\mathbf{n}} \tag{5}$$

where the prime denotes the derivative w.r.t. s. These are the Serret–Frenet equations [36] for a curve $\mathbf{c} : [a, b] \to \mathbb{R}^3$ parametrized by its arclength s. $\kappa = 1/|\mathbf{X}|^2$ is the local curvature of $\mathbf{c}$ and is positive definite, as required, while τ is the local torsion. Because $\mathbf{P}$ is a closed 1-form that is *not* exact, $\mathbf{c}$ is a *closed* loop in $\mathbb{R}^3$ [36].

The existence of solutions to the Serret–Frenet equations is guaranteed [36]. It is nevertheless instructive to look further into their explicit form for two special cases. Let $\kappa = \varpi\cos u$, $\tau = \varpi\sin u$, where $u = u(s)$ and $-\pi/2 \leqslant u \leqslant \pi/2$ because $\kappa \geqslant 0$. Working with the coordinates $\mathrm{d}t = \varpi\,\mathrm{d}s$, equations (5) can be combined into $\ddot{\hat{\mathbf{n}}} = -\hat{\mathbf{n}} + \dot{u}\hat{\mathbf{n}} \times \dot{\hat{\mathbf{n}}}$, where the dot denotes the derivative w.r.t. t. This is similar to the equation of motion for a particle constrained to move on a sphere in the presence of an electric and magnetic dipole (a dyon) at the centre of it, but in this case the ratio of the magnetic to electric 'charge' of the dyon is $\dot{u}$ and can depend on time. Taking $\hat{\mathbf{l}} = \hat{\mathbf{n}} \times \dot{\hat{\mathbf{n}}}$, the torque $\dot{\hat{\mathbf{l}}} = -\dot{u}\dot{\hat{\mathbf{n}}}$ is opposite of the velocity of the particle $\dot{\hat{\mathbf{n}}}$ and has strength $\dot{u}$. Consequently, the total volume $\hat{\mathbf{n}} \cdot \dot{\hat{\mathbf{n}}} \times \ddot{\hat{\mathbf{n}}}$ swept out by $\hat{\mathbf{n}}$ is just $\dot{u}$. If this volume is a constant, then taking $\omega = \sqrt{1 + \dot{u}^2}$,

$$\begin{aligned}
\hat{\mathbf{t}} &= \left\{\cos u\sin(\omega t) - \frac{\dot{u}}{\omega}\sin u\cos(\omega t)\right\}\mathbf{T}_1 \\
&\qquad - \left\{\cos u\cos(\omega t) + \frac{\dot{u}}{\omega}\sin u\sin(\omega t)\right\}\mathbf{T}_2 + \frac{\sin u}{\omega}\mathbf{T}_3 \\
\hat{\mathbf{b}} &= -\left\{\sin u\sin(\omega t) + \frac{\dot{u}}{\omega}\cos u\cos(\omega t)\right\}\mathbf{T}_1 \\
&\qquad - \left\{\sin u\cos(\omega t) - \frac{\dot{u}}{\omega}\cos u\sin(\omega t)\right\}\mathbf{T}_2 + \frac{\cos u}{\omega}\mathbf{T}_3 \\
\hat{\mathbf{n}} &= \frac{1}{\omega}\left\{\cos(\omega t)\mathbf{T}_1 + \sin(\omega t)\mathbf{T}_2 + \dot{u}\mathbf{T}_3\right\}.
\end{aligned} \tag{6}$$

Furthermore, if $\dot{u} = 0$, then $\hat{\mathbf{b}} = \mathbf{T}^3$ and $\mathbf{c}$ is confined to the 1–2 plane. Periodicity of $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ for a closed curve $\mathbf{c}$ gives $2\pi n = \int_a^b \varpi\,\mathrm{d}s = \int_a^b \varpi p$. This is a well-known result [36] for closed curves and is the fundamental reason why p (and consequently $\mathbf{P}$) is a close but not an exact 1-form. For general u, $0 = \dddot{\hat{\mathbf{n}}} - \ddot{u}\ddot{\hat{\mathbf{n}}}/\dot{u} + (1 + \dot{u}^2)\dot{\hat{\mathbf{n}}} - \ddot{u}\hat{\mathbf{n}}/\dot{u}$ with the boundary conditions $|\hat{\mathbf{n}}| = 1$, $|\dot{\hat{\mathbf{n}}}| = 1$ and $\hat{\mathbf{n}}(0) = \mathbf{T}^1$.

4. Vortex rings in $\mathbb{R}^m$

To solve equation (3) for general $\mathfrak{g}$ we follow an approach similar to that in the previous section and introduce a set of linearly independent vectors $\{\hat{\mathbf{t}}_r\} \in \mathbb{R}^m$ on M where $\hat{\mathbf{t}}_r = R_r{}^a\mathbf{T}_a$ such that $\langle\hat{\mathbf{t}}_r, \hat{\mathbf{t}}_s\rangle = \delta_{r,s}$. The set $\{\hat{\mathbf{t}}_r\}$ forms a moving frame on $\mathbb{R}^m$ for points on M (letters in the second half of the alphabet denote coordinates in the moving frame). Then $R_{ra}R_{sa} = \delta_{rs}$ and $R \in so(m)$; similarly, $R_{ra}R_{rb} = \delta_{ab}$. In addition, $[\hat{\mathbf{t}}_r, \hat{\mathbf{t}}_s] = f_{rs}{}^t\hat{\mathbf{t}}_t$, but now $f_{rs}{}^t = R_r{}^a R_s{}^b R^t{}_c f_{ab}{}^c$ are the 'structure constants' in the moving frame. Because they depend on $R_r{}^a$, in this frame $f_{rs}{}^t$ need not be constant.

Since $\{\hat{\mathbf{t}}_r\}$ are orthonormal and span $\mathbb{R}^m$, $\mathrm{d}\hat{\mathbf{t}}_r = -\kappa_{rs}\hat{\mathbf{t}}_s$, where $\kappa_{rs} = -\kappa_{sr}$ are 1-forms on M. Moreover, from $\mathrm{dd}\hat{\mathbf{t}}_r = 0$,

$$0 = \mathrm{d}\kappa_{rs} + \kappa_{rt} \wedge \kappa_{ts}. \tag{7}$$

These are the Maurer–Cartan equations [36] and κ_{rs} are the Maurer–Cartan 1-forms for $so(m)$. Indeed, let $S^{\tilde{a}}$ be the generators of $so(m)$, the symmetry group of $\mathbb{R}^m$, such that $[S^{\tilde{a}}, S^{\tilde{b}}] = k^{\tilde{a}\tilde{b}}_{\tilde{c}} S^{\tilde{c}}$, $\mathrm{Tr}\{S^{\tilde{a}} S^{\tilde{b}}\} = -\delta^{\tilde{a}\tilde{b}}$ and $\tilde{a}$ runs from 1 to $m(m-1)/2$. For a fixed $\tilde{a}$, $S^{\tilde{a}}$ are $m \times m$ antisymmetric matrices with elements $(S^{\tilde{a}})_{rs}$ (we are *not* working on the adjoint representation for $so(m)$). We then introduce the 1-forms $A = A^{\tilde{a}} S^{\tilde{a}}$ with values in the Lie algebra $so(m)$ such that $\kappa_{rs} \equiv (A^{\tilde{a}} S^{\tilde{a}})_{rs}$. Then $\mathrm{d}A + A \wedge A = 0$; A can be seen as a non-Abelian 'vector potential' for the group $so(m)$. The field strength for A vanishes and A is a 'pure gauge' vector potential. As expected, A does not contain any physical degrees of freedom. Indeed, written in terms of matrices of $so(m)$, $R^t\,\mathrm{d}R = -A$.

With this interpretation of the Maurer–Cartan equations it is straightforward to see that

$$\hat{\mathbf{t}}_r = \mathrm{P}\left(\exp\int_0^s A\right)_r^a \mathbf{T}_a \tag{8}$$

where P denotes the Wilson path ordering.

Solution to equation (3) now follows straightforwardly. Given a set of κ_{rs}, we construct $\hat{\mathbf{t}}_r$ using equation (8). We then choose $\mathbf{X} = |\mathbf{X}|\hat{\mathbf{t}}_1$ so that $\mathbf{P} = |\mathbf{X}|^2 f_{1rs}\kappa_{1r}\hat{\mathbf{t}}_s$. Because $\mathrm{d}\mathbf{P} = 0$,

$$0 = \{\mathrm{d}\log|\mathbf{X}|^2 f_{1r's'} - \kappa_{1t'} f_{tr's'}\} \wedge \kappa_{1r'} \tag{9}$$

where $r', s', t' > 1$ and we have used $\mathrm{d}f_{rst} = -\kappa_{rn} f_{nst} - \kappa_{sn} f_{rnt} - \kappa_{tn} f_{rsn}$. In addition, the choice of κ_{rs} must satisfy the constraint $0 = f_{1r's'}\kappa_{1r'} \wedge \kappa_{1s'}$; $\{\kappa_{1r'}\}$ therefore can *not* be linearly independent. One solution of this constraint equation is $\kappa_{1r'} = \kappa_{r'}\pi$, where κ_a are functions on M and π is a 1-form on M. This choice of $\kappa_{1r'}$ does not restrict $\kappa_{r's'}$ and $0 = \mathrm{d}\kappa_{r's'} + \kappa_{r't'} \wedge \kappa_{t's'}$ still. Integration of equation (9) then gives $|\mathbf{X}|^2 = \exp\left\{\int \alpha\pi\right\}$ for any function α on M, and we are done. $\mathbf{X}$ is determined by the arbitrary function α, and Maurer–Cartan 1-forms $\kappa_{r'}\pi$ and $\kappa_{r's'}$.

Except for $so(3)$, $su(2)$ and $sp(2)$, this choice of $\kappa_{1r'}$ is not the most general one that satisfies the constraint equation. Indeed, with this choice, $\mathbf{P}_0 \propto \mathbf{P}_1$ and as the $\mathbb{R}^3$ case, the two components of $\mathbf{P}$ are proportional to one another. It is expected that when the general solution to the constraint equation is used, this relationship between the components of $\mathbf{P}$ will no longer hold.

5. Concluding remarks

We have shown in this paper the deep correlation between I_{RR}, on the one hand, and string theory and TQFT on the other. Indeed, the topological nature of the theory and the fundamental role it plays in determining vortex dynamics are manifest in our approach in analysing the system. Moreover, with this approach generalization of vortex dynamics to higher dimensions becomes straightforward.

Since the goal is to establish the correlations between TQFT, string theory and the study of vortex lines, the approach we have taken in this paper has been purposefully formal. We have focused on establishing mathematical structures and using these structures in understanding the general physical properties of vortices propagating in superfluids that are solely due to the kinetic part of the full Rasette–Regge Lagrangian.

We have focused only on the kinetic part of the Lagrangian for two reasons. First, the traditional interaction term between the vortex lines found in [15] is extremely nonlinear. Some degree of perturbative analysis, based on the kinetic term, would most likely be needed. To this end, a thorough understanding of the 'free' kinetic term is needed. Second, the interaction term has a $1/r$ type of divergence singularity at the classical level, which has traditionally been regulated by introducing a finite vortex core. However, it is expected that the degree of divergence will be weakened in the full quantum field theoretic treatment of the system, and a complete treatment of this divergence will most fruitfully be delayed until then. The first step in this quantum field theoretic approach would be the quantization of the 'free' (kinetic) part of the Rasette–Regge Lagrangian I_{RR}.

How to treat the many-vortex system is still an open question. Once more than one vortex line is introduced, a whole host of questions come to the fore. One particular issue is the question of how interactions between them should be incorporated into the approach outlined here. One can certainly choose to use the classical interaction term found in [15]. Another approach could be to follow the approach of string theory where the interaction of strings is represented by the merging and breaking of strings (which for closed strings fundamentally changes the genus, and thus topology, of the surface it sweeps out). Much of the techniques developed for string theory could then conceivably be applied to the analysis of interacting vortex lines. Which of these two approaches will be more fruitful is unclear, especially in the light of the two points listed above.

The question of how to include interactions between vortices goes beyond a discussion of field-theoretic techniques and methodology, however. As we have mentioned in the introduction, a TQFT has no dynamics in the traditional sense; since the Lagrangian does not depend on the metric, there is no notion of time. Will the inclusion of the interaction terms necessitate the explicit introduction of the metric? While it is possible to use de Rham's method of generalized forms [37] to rewrite and generalize the interaction term found in [15] in terms of differential forms (which will thus automatically be independent of the metric), it is unclear if such an approach is physically meaningful. Moreover, making sense of this interaction term will require the introduction of a high energy cut-off (the vortex core size), which may bring along its own particular set of problems.

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